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From big bang to big crunch and beyond

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Abstract: We study a quotient Conformal Field Theory, which describes a $3+1$ dimensional cosmological spacetime. Part of this spacetime is the Nappi-Witten (NW) universe, which starts at a "big bang" singularity, expands and then contracts to a "big crunch" singularity at a finite time. The gauged WZW model contains a number of copies of the NW spacetime, with each copy connected to the preceeding one and to the next one at the respective big bang/big crunch singularities. The sequence of NW spacetimes is further connected at the singularities to a series of non-compact static regions with closed timelike curves. These regions contain boundaries, on which the observables of the theory live. This suggests a holographic interpretation of the physics.

Keywords: Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Defining observables in quantum gravity in cosmological spacetimes is an interesting problem that received some attention over the years. In particular, in the presence of initial (big bang) and/or final (big crunch) singularities, one might expect that the initial and/or final conditions which give rise to observables must be specified at or near a singularity. This is puzzling since in more familiar situations, such as asymptotically flat spacetimes (with or without a spacelike linear dilaton), and asymptotically AdS spacetimes, boundary conditions (observables) are specified in regions where the theory becomes simple in appropriate variables, whereas near a cosmological singularity the interactions are expected to be strong. It might be that quantum gravity in fact simplifies near such singularities [1], but a satisfactory demonstration of that is currently lacking.

It is natural to enquire whether string theory sheds new light on these issues. The purpose of this paper is to study this question in a particular model introduced by Nappi and Witten [2]. Related four dimensional backgrounds appeared earlier in [3, 4]. They all belong to a moduli space of cosmological string backgrounds with two abelian isometries [5].

The original Nappi-Witten (NW) model describes a four dimensional anisotropic closed universe, which starts from a big bang, and recollapses to a big crunch after a finite time. Therefore, the problem of defining observables is particularly acute in this case [6]. At the same time, the authors of [2] found that this cosmology is described by a certain coset CFT, $[\mathrm{SL}(2)\times\mathrm{SU}(2)]/\,\mathrm{U}(1)^2$, which can be studied using the tools of weakly coupled string theory. In particular, one expects to be able to define observables in the standard fashion, using vertex operators for describing perturbative string states.

It is thus interesting to ask what is the interpretation of the observables one finds in string theory on the NW background in terms of the big bang/big crunch cosmology. String theory resolves this problem in an interesting way. The coset model that gives rise to the NW cosmology describes a spacetime which contains additional regions, which we will refer to as *whiskers* (see figure 3 below). These regions are connected to the cosmological spacetime discussed in [2] at the big bang and big crunch singularities. Moreover, the coset contains a number of copies of the NW spacetime, attached to each other at the respective big bang/big crunch singularities. As we will see below, the different regions are coupled by the dynamics. Thus, this model falls into the pre-big bang class. Other models in this class were discussed e.g. in [7, 8] (for a review, see [9]), and more recently in [10]–[15].

The whiskers are non-compact, time independent and contain a boundary at spatial infinity, near which the background asymptotes to a spacelike linear dilaton one. As in [16], it is natural to define the observables of the model via the behavior of fields near this boundary. The vertex operators describing perturbative strings living on the coset correspond to such observables.

Correlation functions of these observables provide information about physics in the bulk of spacetime. We discuss two classes of observables: those that describe scattering states, which correspond to wavefunctions that are δ-function normalizable near the boundary, and non-normalizable wavefunctions localized at the boundary, that are similar to the standard observables in AdS.

Correlation functions of the δ-function normalizable observables provide information about scattering of states that live near the boundary from the singularities. One finds that incoming waves may be partially reflected back into the non-compact region, and partially transmitted into the NW cosmological region. They might then reemerge in another asymptotic region. Correlation functions of non-normalizable observables provide information about states living in the NW cosmology.

An interesting aspect of the coset CFT is that the whiskers contain closed timelike curves. At first sight this seems to be a major obstacle for including them in the geometry. Nevertheless, they seem to be needed for defining observables, and as we will see later, the dynamics couples the NW cosmological region to the whiskers. It is thus natural to wonder whether the model is physically consistent. This is expected to be the case on general grounds, since the model is obtained by a seemingly sensible gauging of a consistent string background. An observation which might be relevant for this issue is that, technically, the reason for the occurrence of closed timelike curves in the model is the compactness of the SU(2) component of the coset. This compactness is also responsible for the compactness of the NW cosmological region, and for a depletion in the spectrum of perturbative string states. Thus, one may hope that the geometry, spectrum and interactions are precisely such that the model is physically consistent (e.g. free from violations of causality). One of the motivations for this study is to find out whether this is indeed the case.

Another interesting aspect of the physics of the NW model is that the presence of the big bang and big crunch singularities does not appear to lead to a breakdown of string perturbation theory. The correlation functions of vertex operators appear to be well behaved, at least at low orders of string perturbation theory. One possible qualitative

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interpretation of this is that stringy probes are smeared over distances of order the string scale, and thus are able to pass through the big bang and big crunch singularities. Also, it turns out that the NW spacetime is non-Hausdorff near the singularity. This too may play a role in resolving the apparent singularities in the geometry. To shed more light on the behavior near the singularity, it would be interesting to study the dynamics of D-branes in this geometry.

In the remainder of the paper we describe in more detail the structure of the NW model, and make some of the above assertions more precise. The plan of the paper is as follows. In section 2 we discuss the $SL(2,\mathbb{R})$ group manifold. For application to the NW model, it is necessary to diagonalize a spacelike non-compact one dimensional subgroup of $SL(2,\mathbb{R})$. We discuss the geometry and the behavior of eigenfunctions of the above $U(1)$ subgroup. This is a well studied mathematical problem [17]. One finds that the $SL(2,\mathbb{R})$ group manifold splits into different regions. Eigenfunctions of the non-compact spacelike $U(1)$ are generically non-analytic at the boundaries between different regions.

In section 3 we turn to the NW coset. We describe the modifications of the group topology due to the identifications, and the geometrical data corresponding to the coset in different regions. Some regions describe the closed NW cosmology, while others are static and contain asymptotic regimes, closed timelike curves and timelike singularities. We determine the vertex operators of low lying string states corresponding to the principal continuous and discrete series representations of $SL(2,\mathbb{R})$. We present examples of scattering amplitudes which demonstrate that fundamental strings may go from one region to another in the NW background. We also show that the compactness of $SU(2)$ implies a large depletion in the spectrum of physical states in the model.

In section 4 we discuss the quantization of the superstring in the NW background. The theory appears to be non-supersymmetric, but one nevertheless needs to perform a chiral GSO projection. We describe the projection and some aspects of the spectrum of the theory. Our main results are summarized in section 5

2. $SL(2,\mathbb{R})$

2.1 Geometry

The NW model [2] inherits much of its structure from the underlying CFT on $SL(2,\mathbb{R})$. Therefore, in this section we describe some properties of the $SL(2,\mathbb{R})$ group manifold in variables that will be useful when we study the gauged WZW model.

One of the characteristic geometrical features of the coset model will turn out to be its decomposition into two different types of regions. There are regions compact in space and time which can be interpreted as a succession of expanding and contracting universes. There are also non-compact, static regions extending all the way to infinity. This decomposition can be traced back to the geometry of $SL(2,\mathbb{R})$.

A general group element g in $SL(2,\mathbb{R})$ has the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.1}$$

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where

$$ad - bc = 1; \qquad a, b, c, d \in \mathbb{R}. \tag{2.2}$$

The $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ symmetry of the WZW model on the group manifold acts on g via left and right multiplication by independent $\mathrm{SL}(2,\mathbb{R})$ matrices. The NW model is obtained by gauging a group containing the spacelike non-compact $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry generated by σ_3. This acts on g as

$$g \to e^{\alpha\sigma_3} g e^{\beta\sigma_3} = \begin{pmatrix} ae^{\alpha+\beta} & be^{\alpha-\beta} \\ ce^{\beta-\alpha} & de^{-\alpha-\beta} \end{pmatrix}. \tag{2.3}$$

In a unitary representation, eigenfunctions of $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ transform under the action (2.3) as

$$D_{m,\bar{m}} \to e^{2(im\alpha + i\bar{m}\beta)} D_{m,\bar{m}}, \tag{2.4}$$

with real $m, \bar{m}$. Viewed as functions on the group manifold, they have the form

$$D_{m,\bar{m}} = \left(\frac{a}{d}\right)^{\frac{i}{2}(m+\bar{m})} \left(\frac{b}{c}\right)^{\frac{i}{2}(m-\bar{m})} K(ad), \tag{2.5}$$

where K is a function to be determined. Therefore, we see that eigenfunctions of $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ typically exhibit non-analytic behavior when one of the elements of the matrix (2.1) goes to zero. Thus, the $\mathrm{SL}(2,\mathbb{R})$ group manifold naturally splits into different regions, depending on the signs of a, b, c, d. There are twelve different regions:

(A) $ad > 0$, $bc > 0$. There are four such regions, depending on the two signs, $\mathrm{sign}(a) = \mathrm{sign}(d)$ and $\mathrm{sign}(b) = \mathrm{sign}(c)$.

(B) $ad > 0$, $bc < 0$ (four regions).

(C) $ad < 0$, $bc < 0$ (four regions).

The different regions can be distinguished by the value of the quantity

$$W = \mathrm{Tr}\left(\sigma_3 g \sigma_3 g^{-1}\right) = 2(2ad - 1) = 2(2bc + 1), \tag{2.6}$$

which is invariant under the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry (2.3). In regions (A), $W > 2$; in regions (B), $2 > W > -2$; in regions (C), $W < -2$.

In figure 1 we illustrate the resulting structure. The signs indicated in the figure are those of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The regions 1, 1′, 3, 3′ are of type (A) in the classification above, while 2, 2′, 4, 4′ are of type (C). Regions I, II, III, IV are the four regions of type (B). In string theory on AdS_3 (the universal cover of $\mathrm{SL}(2,\mathbb{R})$), a special role is played by the boundary of the $\mathrm{SL}(2,\mathbb{R})$ group manifold, which is the space on which the dual CFT is defined. The boundary corresponds to large a, b, c, d, or $|W| \to \infty$ (see (2.6)). In figure 1 this corresponds to the asymptotic infinities in regions $1 - 4$, $1' - 4'$. The regions I, II, III, IV do not reach the boundary of AdS_3.

After gauging [2], regions of type (B), where $|W| \leq 2$, give rise to NW cosmologies. The others give rise to non-compact, static regions with closed timelike curves.

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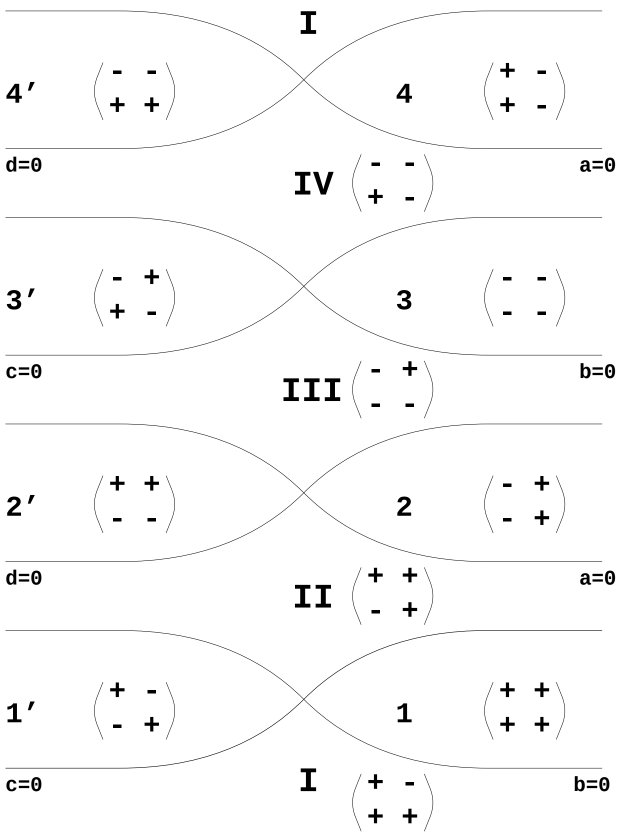

Figure 1: The SL$(2,\mathbb{R})$ group manifold split into regions discussed in the text.

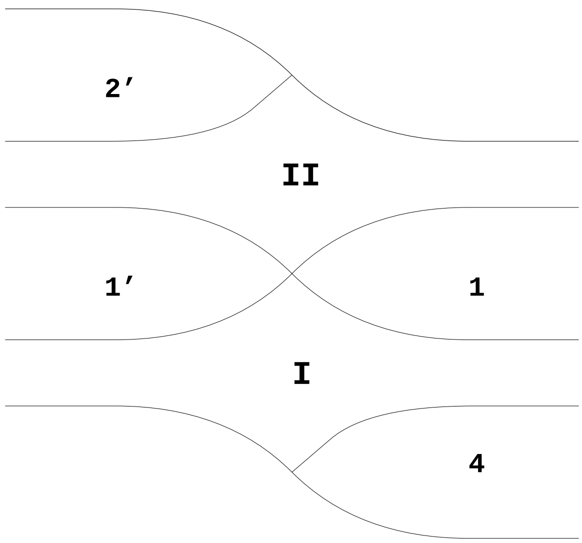

Figure 2: The geometry of PSL$(2,\mathbb{R})$, or the Poincare patch.

One can restrict attention to PSL$(2,\mathbb{R})$, which is obtained by identifying g with $-g$ in (2.1) (a symmetry of (2.2)). Using this symmetry to (say) set $a > 0$, one is left with the six regions 1, $1'$, $2'$, 4, I, II, as indicated in figure 2.

The asymptotic region in $\mathrm{PSL}(2,\mathbb{R})$ (the boundary) is connected, and forms a conformal compactification of two dimensional Minkowski spacetime.

The $\mathrm{PSL}(2,\mathbb{R})$ group manifold as well as its double cover $\mathrm{SL}(2,\mathbb{R})$ are not simply connected; they have closed timelike curves. This is reflected in the periodic identification in the vertical direction in figures 1, 2. This identification is usually avoided by considering the universal cover of the group manifold. An element of the universal cover corresponding to a $g \in \mathrm{SL}(2,\mathbb{R})$ is (the homotopy class of) a curve starting at the identity and ending at g. Since $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \mathbb{Z}$, there is an infinite number of elements in the universal cover corresponding to every $g \in \mathrm{SL}(2,\mathbb{R})$. In the universal cover, the region shown in figure 2 is known as the Poincare patch.

The Poincare patch plays an important role for a number of reasons. First, it is naturally obtained by analytic continuation from euclidean AdS_3. Indeed, the euclidean version of (2.2) is the hyperboloid

$$X_0^2 = X_1^2 + X_2^2 + X_3^2 + 1\,. \tag{2.7}$$

The space (2.7) has two disconnected components, corresponding to positive and negative X_0. Euclidean AdS_3 corresponds to one component. It can be parametrized as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} X_0 + X_1 & X_2 + iX_3 \\ X_2 - iX_3 & X_0 - X_1 \end{pmatrix} = \begin{pmatrix} u & u\gamma^+ \\ u\gamma^- & \frac{1}{u} + u\gamma^+\gamma^- \end{pmatrix}, \tag{2.8}$$

where the Poincare coordinates $u, \gamma^{\pm}$ satisfy $u \in \mathbb{R}_+$, $\gamma^+ \in \mathbb{C}$, and $\gamma^- = (\gamma^+)^*$. The analytic continuation to the lorentzian manifold is obtained by taking γ^+ and γ^- to be independent real (lightlike) variables. The hyperboloid $u \geq 0$ maps to the region $a \geq 0$ in $\mathrm{SL}(2,\mathbb{R})$ (see (2.8)). Since there are no identifications on the coordinates, the resulting lorentzian manifold should be thought of as a patch in the universal cover, and not as $\mathrm{PSL}(2,\mathbb{R})$. This space, the Poincare patch, corresponds to figure 2 without the vertical identifications.

The $\mathrm{SL}(2)$ invariant line element on the Poincare patch is

$$ds^2 = \frac{du^2}{u^2} + u^2 d\gamma^+ d\gamma^-\,. \tag{2.9}$$

Another reason to consider the Poincare patch is that the metric (2.9) is obtained in the near-horizon geometry of branes in string theory. For example, systems of fundamental strings and $NS5$-branes naturally give this geometry (for a review, see [18]).

2.2 Wavefunctions

The vertex operators on the coset are obtained by a restriction of those on the group manifold. Therefore, we will describe in this subsection the explicit forms of eigenfunctions of the laplacian on $\mathrm{SL}(2,\mathbb{R})$ in the basis (2.4), (2.5). We are mainly interested in the Minkowski problem, but since on a euclidean worldsheet the euclidean $\mathrm{AdS}_3 (= H_3^+)$ CFT is better behaved, we first describe the euclidean analogs of these wavefunctions [19].

One starts with the well known eigenfunctions of the laplacian on AdS_3,

$$\Phi_h(x,\bar{x};\phi,\gamma,\bar{\gamma}) = \left(|\gamma - x|^2 e^{\phi} + e^{-\phi}\right)^{-2h}, \tag{2.10}$$

where the coordinates $\phi, \gamma, \bar{\gamma}$ are related to those in (2.8), (2.9) by $u = e^{\phi}$; $\gamma = \gamma^+$; $\bar{\gamma} = \gamma^-$. The corresponding eigenvalue of the laplacian is $-h(h-1)$.

The generators of SL(2) are realized on Φ_h as the differential operators

$$\begin{aligned} J^3 &= \gamma\partial_\gamma - \frac{1}{2}\partial_\phi = -(x\partial_x + h)\,, \\ J^- &= \partial_\gamma = -\partial_x\,, \\ J^+ &= \gamma^2\partial_\gamma - \gamma\partial_\phi - e^{-2\phi}\partial_{\bar{\gamma}} = -(x^2\partial_x + 2hx)\,. \end{aligned} \tag{2.11}$$

Similar formulae hold for the right moving generators $\bar{J}^a$. $\bar{J}^3$ generates the transformation $g \to g e^{\beta\sigma_3}$ in (2.3). This symmetry acts as $u \to e^{\beta}u$, $\gamma \to e^{-2\beta}\gamma$, which corresponds to (holomorphic) dilation symmetry on the boundary of AdS_3. Thus, J^3 can be thought of as L_0 of the Virasoro algebra acting on the boundary of AdS space [20, 21]. In that interpretation, one considers wavefunctions which have $L_0 \in \mathbb{R}$. For our purposes, we saw in eq. (2.4) that one needs to consider wavefunctions with $J^3 \in i\mathbb{R}$. This has to do with the fact that in the analytic continuation from euclidean to Minkowski spacetime corresponding to $X_3 \to iX_3$ in (2.8), one finds that the timelike generator of $\mathrm{SL}(2,\mathbb{R})$ is in fact $-iJ^2 = J^- - J^+$, and $-iJ^3$ becomes a spacelike generator, with real eigenvalues.

Eigenstates of J^3 with imaginary eigenvalues have the (formal) form

$$K_{m,\bar{m};j} = \int d^2x x^{j+im}\bar{x}^{j+i\bar{m}}\Phi_{j+1}(x,\bar{x};\phi,\gamma,\bar{\gamma})\,. \tag{2.12}$$

They are the same as the operators that are usually considered in CFT on AdS_3, with m replaced by im.

We now return to the Minkowski problem, which is richer, since one has to study the wavefunctions in the different regions[1] in figures 1, 2. It is going to be convenient to represent the most general group element $g \in \mathrm{SL}(2,\mathbb{R})$ in the interior of each of the regions[2] as

$$g(\alpha,\beta,\theta;\epsilon_1,\epsilon_2,\delta) = e^{\alpha\sigma_3}(-1)^{\epsilon_1}(i\sigma_2)^{\epsilon_2}g_\delta(\theta)e^{\beta\sigma_3}\,, \tag{2.13}$$

where $\epsilon_1, \epsilon_2 = 0, 1$; $\delta = I, 1, 1'$;

$$g_I = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \qquad 0 \le \theta \le \frac{\pi}{2}\,, \tag{2.14}$$

$$g_1 = g_{1'}^{-1} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}; \qquad 0 \le \theta < \infty\,, \tag{2.15}$$

and σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.16}$$

[1] In euclidean space, comparing (2.5) and (2.8), one finds that a/d, b/c never vanish on H_3^+, and thus there is no analog of the different regions appearing in the Minkowski case.

[2] On the boundaries between the regions one has to use a different representation (we shall return to this later).

Equation (2.13) describes the behavior of the group elements in all twelve regions in figure 1. For example, $\epsilon_1 = \epsilon_2 = 0$ corresponds to the regions I, 1, 1′. In PSL(2, $\mathbb{R}$) and the Poincare patch, ϵ_1 is taken to be identically zero.

Matrix elements of g in a representation with Casimir $-j(j+1)$, $K(j;g)$, are eigenfunctions of the laplacian with eigenvalue $-j(j+1)$. We will discuss two types of unitary representations. Principal continuous representations have

$$j = -\frac{1}{2} + is; \qquad s \in \mathbb{R}, \tag{2.17}$$

and are further labeled by a phase $\exp(i\pi\epsilon)$, where $\epsilon = 0$ in PSL(2, $\mathbb{R}$), $\epsilon = 0, 1$ in SL(2, $\mathbb{R}$), and $\epsilon \in [0, 2)$ for the universal cover. The phase $\exp(i\pi\epsilon)$ corresponds to the representation of the center of the corresponding group. The second class is principal discrete representations, characterized by real j, with

$$j \in \mathbb{Z} + \frac{\epsilon}{2}\,. \tag{2.18}$$

We will choose a basis of eigenvectors of the non-compact U(1), $g = \exp(\alpha\sigma_3)$. For unitary representations, the corresponding eigenvalue is $\exp(2im\alpha)$, with $m \in \mathbb{R}$. In a given representation, m can take any real value. Moreover, for the continuous representations, there are two vectors with the same value of m, which we distinguish by $\pm$.

For the continuous representations in the above basis, the non-vanishing matrix elements of g (2.13) are given by[3]

$$\begin{aligned} K_{\pm\pm}(\lambda,\mu;j,\epsilon;g) &\equiv \langle j,\epsilon,m,\pm|g|j,\epsilon,\bar{m},\pm\rangle = \\ &= e^{2i(m\alpha+\bar{m}\beta)} e^{i\pi\epsilon_1\epsilon} \langle j,\epsilon,m,\pm|(i\sigma_2)^{\epsilon_2} g_\delta(\theta)|j,\epsilon,\bar{m},\pm\rangle\,, \end{aligned} \tag{2.19}$$

where

$$\lambda \equiv -im - j\,; \qquad \mu \equiv -i\bar{m} - j\,. \tag{2.20}$$

These matrix elements appear in [17] (for the group SL(2, $\mathbb{R}$)). We will next give their values in each of the six regions of the Poincare patch. There are two independent functions; we begin in region 1, where we choose them to be K_{++} and K_{--}.[4] They are related via:

$$\begin{aligned} K_{--}(\lambda,\mu;j,\epsilon;g_1) &= K_{++}(-i\bar{m}+j+1,-im+j+1;-(j+1),\epsilon;g_1) = \\ &= \frac{B(-i\bar{m}+j+1,i\bar{m}+j+1)}{B(-im+j+1,im+j+1)} K_{++}(-im+j+1,-i\bar{m}+j+1;-(j+1),\epsilon;g_1)\,. \end{aligned} \tag{2.21}$$

In region 1:

$$K_{++}(\lambda,\mu;j,\epsilon;g_1) = \frac{1}{2\pi i} B(\lambda,-\lambda-2j) \frac{(1-y)^{j+\frac{\lambda+\mu}{2}}}{(-y)^{\frac{\lambda+\mu}{2}}} F\left(\lambda,\mu;-2j;\frac{1}{y}\right)\,, \tag{2.22}$$

[3] We will sometimes use the label g both for the 2×2 matrices (2.1), as well as their representations.

[4] Actually, K_{++} vanishes in region 4 and K_{--} vanishes in region 2' (see below) where one should consider instead K_{-+} or K_{+-}.

$$K_{--}(\lambda,\mu;j,\epsilon;g_1) = \frac{1}{2\pi i}B(1-\mu,\mu+2j+1)\frac{(1-y)^{j+\frac{\lambda+\mu}{2}}}{(-y)^{2j+1+\frac{\lambda+\mu}{2}}}\times$$
$$\times F\left(\lambda+2j+1,\mu+2j+1;2j+2;\frac{1}{y}\right), \tag{2.23}$$
$$y \equiv -\sinh^2\theta\,, \tag{2.24}$$

where g_1 is given in eq. (2.15), λ,μ are given in (2.20), $B(a,b)$ is the Euler Beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,, \tag{2.25}$$

and $F(a,b;c;x)$ is the hypergeometric function ${}_2F_1$.

In region $1'$ one finds

$$K_{++}(\lambda,\mu;j,\epsilon;g_{1'}) = K_{--}(\lambda,\mu;j,\epsilon;g_1)\,, \tag{2.26}$$
$$K_{--}(\lambda,\mu;j,\epsilon;g_{1'}) = K_{++}(\lambda,\mu;j,\epsilon;g_1)\,, \tag{2.27}$$

where $g_{1'} = g_1^{-1}$ is given in eq. (2.15), and $K_{++}(g_1), K_{--}(g_1)$ are given in eqs. (2.22), (2.23), respectively.

In region I:

$$K_{++}(\lambda,\mu;j,\epsilon;g_I) = \frac{1}{2\pi i}B(\lambda,\mu+2j+1)\frac{(-x)^{j+\frac{\lambda+\mu}{2}}}{(1-x)^j}F(\lambda,\mu;\lambda+\mu+2j+1;x)\,, \tag{2.28}$$
$$K_{--}(\lambda,\mu;j,\epsilon;g_I) = \frac{1}{2\pi i}B(1-\mu,-\lambda-2j)\frac{(1-x)^{j+1}}{(-x)^{j+\frac{\lambda+\mu}{2}}}F(1-\lambda,1-\mu;1-\lambda-\mu-2j;x)\,, \tag{2.29}$$
$$x \equiv -\,\mathrm{ctg}^2\,\theta\,, \tag{2.30}$$

where g_I is given in eq. (2.14).

In region II one has:

$$K_{++}(\lambda,\mu;j,\epsilon;g_{II}) = K_{--}(\lambda,\mu;j,\epsilon;g_I)\,, \tag{2.31}$$
$$K_{--}(\lambda,\mu;j,\epsilon;g_{II}) = K_{++}(\lambda,\mu;j,\epsilon;g_I)\,, \tag{2.32}$$

where $g_{II} = (g_I)^{-1}$ (g_I is given in eq. (2.14)), and $K_{++}(g_I)$, $K_{--}(g_I)$ are given in eqs. (2.28), (2.29), respectively.

In region $2'$:

$$K_{++}(\lambda,\mu;j,\epsilon;g_{2'}) = K_{-+}(-\lambda-2j,\mu;j,\epsilon;g_1)\,, \tag{2.33}$$
$$K_{--}(\lambda,\mu;j,\epsilon;g_{2'}) = K_{+-}(-\lambda-2j,\mu;j,\epsilon;g_1) = 0\,, \tag{2.34}$$

where $g_{2'} = i\sigma_2 g_1$ (g_1 is given in eq. (2.15)), and

$$K_{-+}(\lambda,\mu;j,\epsilon;g_1) = \tag{2.35}$$
$$= \frac{1}{2\pi i}(1-y)^{j+\frac{\lambda+\mu}{2}}\Big[B(\lambda,1-\mu)(-y)^{\frac{\lambda-\mu}{2}}F(\lambda,\lambda+2j+1;\lambda-\mu+1;y)+$$
$$+(-)^\epsilon B(-\lambda-2j,\mu+2j+1)(-y)^{\frac{\mu-\lambda}{2}}F(\mu,\mu+2j+1;\mu-\lambda+1;y)\Big],$$

with y given in eq. (2.24). Here and below the formulae are valid for PSL(2) where $\epsilon = 0$ as well as SL(2) where ϵ is either 0 or 1.

Finally, in region 4 one finds:

$$K_{++}(\lambda, \mu; j, \epsilon; g_4) = (-)^{\epsilon} K_{+-}(-\lambda - 2j, \mu; j, \epsilon; g_1) = 0\,, \tag{2.36}$$

$$K_{--}(\lambda, \mu; j, \epsilon; g_4) = (-)^{\epsilon} K_{-+}(-\lambda - 2j, \mu; j, \epsilon; g_1)\,, \tag{2.37}$$

where $g_4 = -i\sigma_2(g_1)^{-1} = -g_1 i\sigma_2$ (g_1 is given in eq. (2.15)), and $K_{-+}(g_1)$ is given in eq. (2.35).

The behavior of the wavefunctions on the two dimensional surfaces separating the various regions, i.e., $g \in \mathrm{SL}(2,\mathbb{R})$ one of whose matrix elements is equal to zero, requires a special treatment. Any $\mathrm{SL}(2,\mathbb{R})$ matrix with a vanishing entry can be written as

$$g = e^{\phi\sigma_3}(-1)^{\epsilon_1}(i\sigma_2)^{\epsilon_2} g_\gamma (i\sigma_2)^{\epsilon_3}\,, \tag{2.38}$$

where $\epsilon_{1,2,3} = 0, 1$;

$$g_\gamma = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}; \qquad \gamma > 0\,. \tag{2.39}$$

The wavefunctions on all the "lines" in figure 1 can thus be obtained from, say,

$$K_{++}(\lambda, \mu; j, \epsilon; g_\gamma) = K_{--}(\lambda, \mu; j, \epsilon; g_\gamma^{-1}) = \frac{1}{2\pi i} B(\lambda, \mu - \lambda)\gamma^{\lambda-\mu}\,. \tag{2.40}$$

Note that as $\varepsilon \equiv \lambda - \mu = i(\bar{m} - m) \to 0$, $K \sim \gamma^{\varepsilon}/\varepsilon$. This indicates a logarithmic divergence of the wavefunctions on the boundary $b = 0$ (and similarly, on the other boundaries between the various regions the wavefunctions are logarithmically divergent either when $m = \bar{m}$ or $m = -\bar{m}$).

A particularly interesting wavefunction is the combination [22]:

$$\mathrm{U}(\lambda, \mu; j, \epsilon; g) \sim K_{++} - \frac{\sin(\pi\mu)}{\sin(\pi\lambda)} K_{--}\,. \tag{2.41}$$

It is infinitely blue shifted as $b \to 0$ (the boundary of region 1): $\mathrm{U}(b \to 0) \sim b^{i(m-\bar{m})}$, and hence any normalizable wave packet constructed as a superposition of U's with different values of m and $\bar{m}$ vanishes at $b = 0$. Its asymptotic behavior in region 1 (as $\theta \to \infty$) is:

$$\mathrm{U}(\chi, \omega; j; \theta \to \infty) \sim e^{2i\chi\phi} e^{-\theta} \left[e^{2i(\omega t + s\theta)} + R(j; m, \bar{m}) e^{2i(\omega t - s\theta)} \right], \tag{2.42}$$

where j and s are related by (2.17),

$$t = \alpha - \beta\,, \qquad \phi = \alpha + \beta\,, \qquad \omega = \frac{1}{2}(m - \bar{m})\,, \qquad \chi = \frac{1}{2}(m + \bar{m})\,, \tag{2.43}$$

and

$$R(j; m, \bar{m}) = \frac{\Gamma(-2j-1)\Gamma(j+1+im)\Gamma(j+1-i\bar{m})}{\Gamma(2j+1)\Gamma(-j+im)\Gamma(-j-i\bar{m})}\,. \tag{2.44}$$

For $\omega, s > 0$, the combination U looks like an incoming plane wave in region 1, scattered from the line $b = c = 0$ (where regions $1, 1', I$ and II intersect). The damping factor $e^{-\theta}$ is canceled by a corresponding factor in the $SL(2,\mathbb{R})$ measure. $R(j;m,\bar{m})$ is the "reflection coefficient" of a plane wave coming in from the boundary.[5] It is also equal to the two point function of $V_{j;im,i\bar{m}}$, a primary field of $SL(2)_L \times SL(2)_R$ with "spin" j in the $SL(2)_k$ WZW model in the large k limit (see [23, eq. (3.6)]). Note also that the reflection coefficient is a phase when $m = \bar{m}$. This seems to be related to the fact that for $m \neq \bar{m}$, the wavefunction (2.41) is non-analytic all along $b = 0$; this behavior allows part of the wave to leak from region 1 to region II through the $b = c = 0$ line. For $m = \bar{m}$, each of the wavefunctions K_{++} and K_{--} separately has a logarithmic singularity at $b = c = 0$, but the combination U is regular, leading to complete reflection from the line $b = c = 0$.

In string theory on AdS_3, principal continuous representations actually describe "bad tachyons" which are absent in infrared stable vacua.[6] "Good tachyons" and massive states correspond to principal discrete representations, with real j. The corresponding normalizable wavefunctions are given by particular linear combinations of the $K_{\pm\pm}$ above. Explicitly, in region 1, the normalizable wavefunctions are

$$K_{++}(g_1) \qquad \text{if} \qquad j < -\frac{1}{2}\,, \tag{2.45}$$

$$K_{--}(g_1) \qquad \text{if} \qquad j > -\frac{1}{2}\,, \tag{2.46}$$

where $K_{++}(g_1)$ and $K_{--}(g_1)$ are given in eqs. (2.22) and (2.23), respectively, with $j \in \mathbb{R}$. The continuation of these functions to the other regions is normalizable as well iff j and ϵ are correlated as in eq. (2.18). Actually, the functions in eqs. (2.45), (2.46) are not independent, due to the relation (2.21). This is in agreement with the familiar fact that only one of the two independent solutions of the Laplace equation on AdS_3 — the universal cover of $SL(2)$ — is normalizable. These normalizable wavefunctions decay exponentially towards the boundary of AdS_3. Their counterparts — observables in string theory — are exponentially supported at the boundary, a well known fact in string theory on AdS [18].

3. The coset

3.1 Geometry

The string theory background we are interested in has the form

$$\left[SL(2,\mathbb{R}) \times \frac{SU(2)}{U(1)} \times U(1)\right] \times \mathcal{M}_6\,, \tag{3.1}$$

where $\mathcal{M}_6$ is a compact manifold, say $\mathcal{M}_6 = T^6$. We will mostly discuss the four-

[5] The reference to "incoming wave," "outgoing wave" and "reflection" is more appropriate for discussing the NW coset, to which we turn in the next section.

[6] Representations with $j = -1/2 + is$ are also relevant for the description of certain "long string" states [24], but we are not going to discuss them here.

dimensional (coset) part of the geometry, which can be described as follows. Let (g, g') be a point in the product manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$. We identify all the points on the $\mathrm{U}(1) \times \mathrm{U}(1)$ orbit parametrized by (ρ, τ),

$$(g, g') \to \left(e^{\rho\sigma_3} g e^{\tau\sigma_3}, e^{i\tau\sigma_3} g' e^{i\rho\sigma_3}\right) . \tag{3.2}$$

This is a special case of [2], who discussed a one parameter set of coset CFT's labeled by a parameter α. Equation (3.2) corresponds to $\alpha = 0$ in the notation of that paper.

Parametrizing the $\mathrm{SL}(2,\mathbb{R})$ matrix g as in (2.13), and similarly expressing the SU(2) matrix g' in terms of Euler angles

$$g'(\alpha', \beta', \theta') = e^{i\beta'\sigma_3} e^{i\theta'\sigma_2} e^{i\alpha'\sigma_3} , \tag{3.3}$$

the gauge transformation (3.2) takes the form

$$\begin{aligned} \alpha &\to \alpha + \rho \\ \beta &\to \beta + \tau \\ \alpha' &\to \alpha' + \rho \\ \beta' &\to \beta' + \tau \\ \theta &\to \theta \\ \theta' &\to \theta' \\ \epsilon_1, \epsilon_2, \delta &\to \epsilon_1, \epsilon_2, \delta . \end{aligned} \tag{3.4}$$

The points of the coset manifold are the orbits of the $\mathrm{SL}(2) \times \mathrm{SU}(2)$ manifold under (3.4). To get a picture of the structure of the coset manifold one can fix the gauge by setting the two SL(2) coordinates α and β to zero. This leaves four coordinates, the three compact coordinates α', β' and θ' of SU(2) and the non-compact θ together with the discrete coordinates ϵ_1, ϵ_2 and δ, surviving from SL(2). From this point of view the coset manifold is a continuous family of SU(2) manifolds, topologically three-spheres, depending on the parameter θ, for each of the regions described in the previous section. Algebraically, this gauge corresponds to viewing the NW universe as a θ-dependent $J_3\bar{J}_3$ deformation of the SU(2) WZW model [25] (for a review, see [26]).

This description breaks down for $\theta = 0$ or $\theta = \pi/2$ if $\delta = I$, where different regions meet. The SL(2) matrices corresponding to these values of θ are fixed by some U(1) subgroup of (2.3) and cannot be used for a complete gauge fixing. At these points in SL(2), part of the gauge fixing has to be imposed on the SU(2) part, so the three-spheres which sit above $\theta = 0$ or $\theta = \pi/2$ for $\delta = I$, are twisted by gauge identifications.

We will use an alternative gauge fixing, $\alpha' = \beta' = 0$. In this gauge, the coset is labeled by the SL(2) parameters α, β, θ and the discrete labels $\epsilon_1, \epsilon_2, \delta$, as well as the (compact) SU(2) coordinate $\theta' \in [0, \pi/2]$. Each $0 < \theta' < \pi/2$, for which (3.3) is a good parametrization, corresponds to a copy of the full SL(2) manifold.

It is important to note that the gauge condition $\alpha' = \beta' = 0$ does not fix the gauge completely. In the parametrization (3.3) of SU(2), $\alpha' + \beta'$ and $\alpha' - \beta'$ are only defined modulo 2π. Thus the transformations (3.4) with $\rho + \tau = 2\pi n_1, \rho - \tau = 2\pi n_2$, with

$n_1, n_2 \in \mathbb{Z}$, preserve the gauge conditions on α' and β': a residual $\mathbb{Z} \times \mathbb{Z}$ gauge symmetry has survived the gauge fixing. This implies that in the SL(2) copy sitting above each θ' in the interval $(0, \pi/2)$, a further identification has to be made under (3.4) with $\rho + \tau = 2\pi n_1$, $\rho - \tau = 2\pi n_2$. In the parametrization (2.1) of SL(2), this identification reads (see (2.3)),

$$\begin{aligned} a &\to a e^{2\pi n_1} \\ b &\to b e^{2\pi n_2} \\ c &\to c e^{-2\pi n_2} \\ d &\to d e^{-2\pi n_1} . \end{aligned} \tag{3.5}$$

Notice that the points on the line $a = d = 0$ are fixed under the $\mathbb{Z}$ subgroup corresponding to $n_2 = 0$, while those on the line $b = c = 0$ are preserved by the $\mathbb{Z}$ subgroup $n_1 = 0$. As a result, we expect an orbifold singularity of the coset manifold at these surfaces. Note that each point with $b = 0$ or with $c = 0$ is identical by (3.5) to a point arbitrarily close to $b = c = 0$. Similarly for a and d.

Taking into account the identification (3.5), the physical space corresponds to a fundamental domain resulting from the division of the SL(2) manifold, sitting above each $\theta' \in (0, \pi/2)$, by (3.5). Such a fundamental domain in SL(2), for group elements corresponding to matrices g (2.1) with non-zero entries a, b, c, d, can be chosen as the region

$$\begin{aligned} 1 \geq \left|\frac{b}{c}\right| &> e^{-4\pi} \\ 1 \geq \left|\frac{a}{d}\right| &> e^{-4\pi} . \end{aligned} \tag{3.6}$$

To that one should add two regions with $b = 0$ and $|c|$ arbitrarily small together with two similar intervals with $c = 0$ and $|b|$ arbitrarily small. Two additional pairs of such regions should be added in the neighborhood of the lines $a = 0$ and $d = 0$. These regions form the fundamental domain for the set of SL(2) matrices with one vanishing element. The identification results in a non-Hausdorff manifold, see e.g. [27].

The structure shown in figure 1 for $SL(2, \mathbb{R})$ turns after the identification (3.5) into that shown in figure 3. Spacetime consists of a sequence of closed NW cosmologies connected at the singularities, where they are also attached to additional regions which were referred to above as whiskers. Near the big bang/big crunch singularities, the manifold is non-Hausdorff. Near the line $b = c = 0$, the fundamental domain (3.6) has the form of four three-dimensional wedges meeting at the line. It has there the form of a (b, c) plane divided by a finite boost (a Misner universe [28]), times a finite interval in the coordinate a.[7] Similarly, near the line $a = d = 0$ the modded out SL(2) manifold looks like the (a, d) plane divided by a finite boost times a finite interval in b. This three dimensional manifold sits above each point $\theta' \in (0, \pi/2)$, thus forming a four dimensional universe.

To get the string frame metric, antisymmetric tensor and dilaton on this manifold, one starts from the SL(2) × SU(2) sigma model and introduces two U(1) gauge fields A^ρ and A^τ corresponding to the two U(1) identifications in (3.2). As usual [29], the geometrical data

[7] Misner background in string theory appeared also in [10].

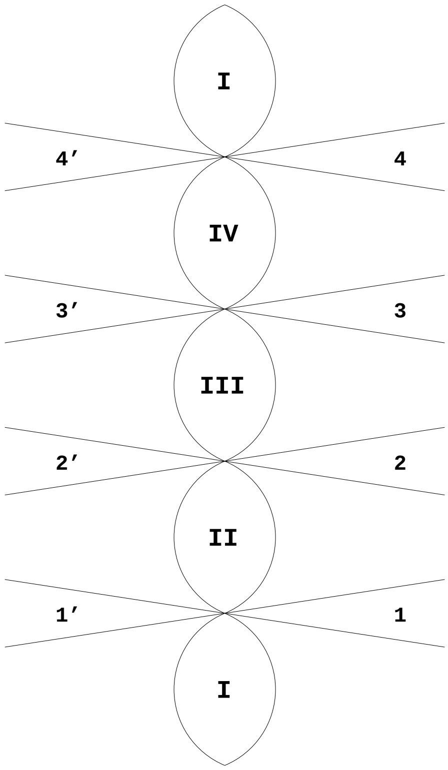

Figure 3: A two dimensional slice of the four dimensional coset spacetime. In the "closed universes" — regions I, II, III, IV — time (θ) runs vertically. The horizontal axis represents $\lambda_\pm$, θ'. In the "whiskers" — regions $1-4$, $1'-4'$ — θ is spacelike and corresponds to the horizontal axis. Time is either λ_+ or λ_-, depending on the value of θ' (see figure 4).

of the coset manifold is obtained by fixing the gauge and integrating out these gauge fields. We use the coordinates (2.13) and (3.3) for $SL(2)\times SU(2)$ and fix the gauge $\alpha'=\beta'=0$.

In regions I, III, corresponding to $|W|<2$, with W defined in (2.6), the procedure described above gives

$$\frac{1}{k}ds^2 = -d\theta^2 + d\theta'^2 + \frac{\cot^2\theta'}{1+\tan^2\theta\cot^2\theta'}d\lambda_-^2 + \frac{\tan^2\theta}{1+\tan^2\theta\cot^2\theta'}d\lambda_+^2 \tag{3.7}$$

$$B_{\lambda_+,\lambda_-} = \frac{k}{1+\tan^2\theta\cot^2\theta'} \tag{3.8}$$

$$\Phi = \Phi_0 - \frac{1}{2}\log\left(\cos^2\theta\sin^2\theta' + \sin^2\theta\cos^2\theta'\right) \tag{3.9}$$

where $\alpha\pm\beta\equiv\lambda_\pm\in[0,2\pi)$, and θ and θ' vary in the interval $[0,\pi/2]$. In regions II, IV, λ_+ and λ_- in (3.7) switch their roles. The parameter k determines the maximal size of the universe (which is $\sqrt{k}l_s$). In the CFT corresponding to the background (3.1), k is the level of SL(2) and SU(2) (see section 4).[8] The dilaton Φ is normalized such that the string coupling is $g_s=e^\Phi$.

[8]The geometric data obtained is valid in the large k limit. For the bosonic string there are known $1/k$ corrections [30]. The exact background sometimes has a different singularity structure [31]. For fermionic strings, the semiclassical background is expected to be a solution to all orders in $1/k$ [30, 32].

For the external regions corresponding to $W > 2$ we find

$$\frac{1}{k}ds^2 = d\theta^2 + d\theta'^2 + \frac{\cot^2\theta'}{1-\tanh^2\theta\cot^2\theta'}d\lambda_+^2 - \frac{\tanh^2\theta}{1-\tanh^2\theta\cot^2\theta'}d\lambda_-^2 \tag{3.10}$$

$$B_{\lambda_+,\lambda_-} = \frac{k}{1-\tanh^2\theta\cot^2\theta'} \tag{3.11}$$

$$\Phi = \Phi_0 - \frac{1}{4}\log\left(\cosh^2\theta\sin^2\theta' - \sinh^2\theta\cos^2\theta'\right)^2 \tag{3.12}$$

where here $0 \leq \theta < \infty$, $0 \leq \theta' \leq \pi/2$, $\lambda_\pm \in [0, 2\pi)$. For $W < -2$, λ_+ and λ_- in (3.10) switch their roles.

The coordinates in (3.7), (3.8) and (3.9) cover each of the internal regions in figure 3. Those of (3.10), (3.11) and (3.12) cover an external region in the figure. In the internal regions, those with $|W| < 2$, (3.7) implies that θ is a timelike coordinate varying over the finite interval $[0, \frac{\pi}{2}]$. In the external regions θ becomes spacelike; the timelike coordinate there is either λ_- or λ_+ depending on θ and θ' (see figure 4). Since the coordinates $\lambda_\pm$ are periodic, the external regions in figure 3 — the "whiskers" — contain closed timelike curves. Note also that the whiskers correspond to a time-independent background. The scalar curvature in the whiskers is non-positive. On the other hand, in the compact parts of spacetime the sign of the scalar curvature is position dependent.

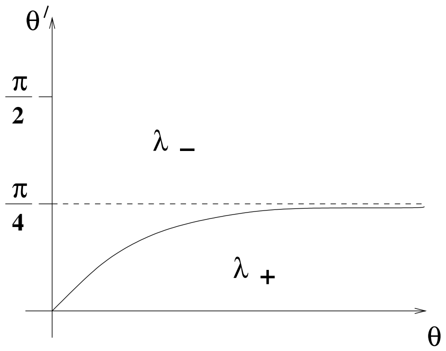

Figure 4: The time coordinate in the metric (3.10) valid in regions $1-4$, $1'-4'$ depending on the value of θ and θ'. Note that for $\theta' > \pi/4$, λ_- is the time coordinate for all values of θ. For $\theta' < \pi/4$, λ_- serves as the time coordinate for $\cosh 2\theta < 1/\cos 2\theta'$.

As mentioned in section 2, before gauging the $\mathrm{U}(1)\times\mathrm{U}(1)$, the boundary of AdS_3 corresponds to large $|W|$ (2.6); after gauging, this corresponds to $\theta \to \infty$ in (3.10) – (3.12). In this limit the background factorizes. θ is described by an asymptotically linear dilaton CFT; it is natural [16] to interpret $\theta \to \infty$ as a holographic screen. It is labeled by $\{\theta', \lambda_\pm\}$, which form a three dimensional spacetime with metric, B-field and dilaton given by:

$$\begin{aligned}
\frac{1}{k}ds^2 &= d\theta'^2 + \frac{\cot^2\theta'}{1-\cot^2\theta'}d\lambda_+^2 - \frac{1}{1-\cot^2\theta'}d\lambda_-^2 \\
B_{+-} &= \frac{k}{1-\cot^2\theta'} \\
\Phi &= \Phi_0 - \frac{1}{4}\log(\cos 2\theta')^2 \,.
\end{aligned} \tag{3.13}$$

This spacetime contains a timelike singularity at $\theta' = \pi/4$, a kind of domain wall.

In an internal region, when the timelike coordinate θ tends to 0, the spatial metric in (3.7) shrinks to zero volume due to the vanishing of the $d\lambda_+^2$ term. Similarly when θ tends to $\pi/2$, the coefficient of $d\lambda_-^2$ vanishes. Such an internal region was interpreted in [2] as a four dimensional universe starting from a big bang at time $\theta = 0$ and ending at a big crunch at $\theta = \pi/2$. Figure 3 describes then a series of closed universes; the big bang of each of them is the big crunch of the previous one. Each such cosmological region is connected at the singularity to two non-compact, time independent external regions with closed timelike curves. In the universal cover of $\mathrm{SL}(2, \mathbb{R})$, the geometry contains an infinite number of copies of the structure exhibited in figure 3.

When $\theta = \theta' = 0$, the metric (3.7) develops a curvature singularity and the dilaton (3.9) becomes singular. This corresponds in $\mathrm{SL}(2) \times \mathrm{SU}(2)$ to both g and g' in (3.2) being proportional to the identity matrix. This is a fixed point of a U(1) subgroup of (3.2) corresponding to $\rho = -\tau$. Points preserved by a subgroup of the gauge group give rise to singularities in the coset manifold. Similarly, (3.7) and (3.9) are also singular at $\theta = \theta' = \pi/2$. In $\mathrm{SL}(2) \times \mathrm{SU}(2)$ this is the point corresponding to both g and g' proportional to the matrix $i\sigma_2$. This is fixed by the subgroup of (3.2) with $\rho = \tau$.

What attitude should one take towards the blowing up of the curvature and coupling constant at some points? In general relativity this is a problem. On the other hand, a string theory based on a coset model is defined algebraically by a perturbative genus expansion. The correlation functions are inherited from those of the group manifold, which are regular.[9] Hence strings can avoid the classical pathology, as might be expected for extended objects, even in a singular background [33] (for a review, see [26]). A familiar example of this phenomenon is the geometric description of the $\mathrm{SU}(2)/\mathrm{U}(1)$ parafermion CFT as a bell-shaped manifold [34] where both curvature and dilaton blow up at the edge. In that case it is clear that string theory on $\mathrm{SU}(2)/\mathrm{U}(1)$ is perturbatively non-singular.

The background (3.7)–(3.12) has two abelian isometries. As such, it is a particular point in the moduli space of more general backgrounds related by the action of $O(2,2,\mathbb{R})$ rotations [5] (for a review, see [26]). A simple point in this moduli space corresponds to the direct product of an $\mathrm{SU}(2)/\mathrm{U}(1)$ parafermion sigma model with the lorentzian $\mathrm{SL}(2)/\mathrm{U}(1)$ two dimensional black hole. Each closed NW cosmological region denoted by a roman numeral in figure 3 is mapped by $O(2,2,\mathbb{R})$ to a region between the horizon and singularity of the two dimensional black hole. The whiskers are mapped to the regions outside the horizon and behind the singularity.

The discrete $O(2,2,\mathbb{Z})$ subgroup of $O(2,2,\mathbb{R})$ acts on this moduli space as a T-duality group. In particular, there is an interesting T-duality transformation which does not change the global shape of the NW universe but, locally, removes the singularities at $\theta = \theta' = 0$ and $\theta = \theta' = \pi/2$, creating instead singularities at $\theta = 0$, $\theta' = \pi/2$ and $\theta = \pi/2$, $\theta' = 0$.[10] This is another reason to believe that the physics of the NW model is non-singular at $\theta = \theta' = 0,\ \pi/2$.

[9]More precisely, this is true for the unintegrated correlation functions, which are the same in the CFT on the group manifold and in the coset CFT, up to known functions of the worldsheet positions (see e.g. [23]). Thus, two and three point functions are the same in the group manifold and in the coset CFT, while for higher correlation functions, the integrated correlation functions are in general different.

[10]This is similar to what happens upon T-duality in the $\mathrm{SU}(2)/\mathrm{U}(1)$ sigma model; actually, at the direct product point discussed above this T-duality is the one acting on the parafermion piece.

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As mentioned earlier, the compact universe (3.7)–(3.9) can be described as a time-dependent $J\bar{J}$ deformation of an SU(2) WZW model. In particular, the T-duality transformation which maps the deformation line of the $\mathrm{SU}(2)_k$ WZW model to $[\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \times S^1_{\sqrt{k}R}]/\mathbb{Z}_k$ (see [35] for details), takes the cosmological CFT into an orbifold of the product of a parafermion with a time dependent circle, with radius $R(\theta) = \tan\theta$. This equivalent CFT description might be useful for exploring the theory further.[11]

For the external regions, there appears in (3.10) – (3.12) a singular surface consisting of points with

$$\tan\theta' = \tanh\theta\,. \tag{3.14}$$

These do not correspond to any fixed points on the group manifold. To understand the origin of this singularity recall that the dynamics on the group manifold is governed by the quadratic form $E = G + B$ where G is the metric and B the antisymmetric tensor. The group manifold is then modded out by identifying all the points on the orbit of the gauge group which is a two dimensional surface in the group manifold parametrized by ρ and τ of (3.2). The singular points in (3.14) correspond to the orbits for which the quadratic form E induced from the full group manifold on the orbit becomes degenerate. Indeed, the metric on the six dimensional $\mathrm{SL}(2) \times \mathrm{SU}(2)$ manifold, in the region of SL(2) with $W > 2$, is

$$ds^2 = d\theta^2 + \cosh^2\theta d\lambda_+^2 - \sinh^2\theta d\lambda_-^2 + d\theta'^2 + \cos^2\theta' d\lambda_+'^2 + \sin^2\theta' d\lambda_-'^2\,, \tag{3.15}$$

where the coordinates (2.13) and (3.3) are used with the substitution $\lambda_\pm = \alpha \pm \beta$ and $\lambda'_\pm = \beta' \pm \alpha'$. For $W < -2$, λ_+ and λ_- in (3.15) switch roles. The induced metric on a (ρ, τ) orbit is gotten from (3.15), using (3.4), by substituting

$$\begin{aligned} d\lambda_+ &= d\rho + d\tau \\ d\lambda_- &= d\rho - d\tau \\ d\lambda'_+ &= d\rho + d\tau \\ d\lambda'_- &= d\tau - d\rho\,. \end{aligned} \tag{3.16}$$

The induced metric on the orbit is then

$$ds^2 = 2\left[d\rho^2 + d\tau^2 + \left(\cosh 2\theta + \cos 2\theta'\right) d\rho d\tau\right]. \tag{3.17}$$

Notice that (unlike the regions with $|W| < 2$) this gauged surface is not spacelike for large θ.

The Wess-Zumino three-form for the $\mathrm{SL}(2) \times \mathrm{SU}(2)$ group is, for $|W| > 2$,

$$\frac{1}{3}\,\mathrm{Tr}\left(g^{-1}dg\right)^3 = \frac{1}{2}(\sinh 2\theta\, d\lambda_+ \wedge d\lambda_- \wedge d\theta - \sin 2\theta'\, d\lambda'_+ \wedge d\lambda'_- \wedge d\theta')\,. \tag{3.18}$$

[11] In the vicinity of $\theta = 0$, the CFT background $[\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \times S^1_{\sqrt{k}R(\theta)} \times \mathbb{R}_\theta]/\mathbb{Z}_k$ (where $\mathbb{R}_\theta$ is timelike) behaves like $[\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \times \frac{\mathbb{R}^{1,1}}{\mathbb{Z}}]/\mathbb{Z}_k$. This is the product of a two dimensional Misner universe with a two dimensional parafermion sigma model (modded by a $\mathbb{Z}_k$ which acts as a further boost in $\mathbb{R}^{1,1}/\mathbb{Z}$ together with twisting the $\mathbb{Z}_k$ parafermion CFT).

This gives for the B field on the group manifold:

$$B = \cosh 2\theta \, d\lambda_+ \wedge d\lambda_- + \cos 2\theta' \, d\lambda'_+ \wedge d\lambda'_- \,. \tag{3.19}$$

Substituting (3.16), the induced B field on the gauge orbit is

$$B = 2(\cos 2\theta' - \cosh 2\theta) \, d\rho \wedge d\tau \,. \tag{3.20}$$

The induced quadratic form $E = G + B$ on the gauge orbit is the sum of (3.17) and (3.20). In the (ρ, τ) coordinates it takes the form,

$$E = 2 \begin{pmatrix} 1 & \cos 2\theta' \\ \cosh 2\theta & 1 \end{pmatrix} . \tag{3.21}$$

This form degenerates when

$$\cosh 2\theta \, \cos 2\theta' = 1 \,, \tag{3.22}$$

which is the same as condition (3.14) which determines the singularities in the external regions of the coset manifold.

3.2 Wavefunctions

Rather than studying the system as a sigma model describing string motion on the complicated singular manifold presented in the previous subsection, we will try to make use of its representation [2] as a quotient of the much smoother $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SU}(2)$ group manifold. The first step is to identify vertex operators on the group manifold, which are invariant under the gauge identification. These give rise to vertex operators in the quotient theory. A typical unexcited vertex operator on $\mathrm{SL}(2) \times \mathrm{SU}(2)$ is of the form

$$V^{j,j'}_{m,m';\bar{m},\bar{m}'} = K^j_{m,\bar{m}}(g) D^{j'}_{m',\bar{m}'}(g') \,, \tag{3.23}$$

where $K^j_{m,\bar{m}}(g)$ is a matrix element representing $g \in \mathrm{SL}(2)$ in the unitary representation j between vectors labeled by m and $\bar{m}$. $D^{j'}_{m',\bar{m}'}(g')$ is similarly defined for $\mathrm{SU}(2)$. For the gauging (3.2), it will be convenient to choose for both the $\mathrm{SL}(2)$ and $\mathrm{SU}(2)$ representations a basis in which the $\mathrm{U}(1)$ subgroup generated by σ_3 is diagonal. As in section 2, a vector labeled by m is an eigenvector of the operator representing the element $e^{\sigma_3 \rho}$ of $\mathrm{SL}(2)$, with eigenvalue $e^{2im\rho}$. A vector labeled by m' has the eigenvalue $e^{2im'\rho}$ for the operator representing the element $e^{i\sigma_3 \rho}$ of $\mathrm{SU}(2)$. As described in the previous section, for $j = -1/2 + is$, corresponding to a continuous series representation of $\mathrm{SL}(2)$, the representation is not fully determined by j but rather depends on an additional parameter ϵ. Also, in that case, the labels m and $\bar{m}$ do not fully specify a vector in the representation — one needs to further specify a Z_2 valued index denoted above by $\pm$. Both $K^j_{m,\bar{m}}(g)$ and $V^{j,j'}_{m,m';\bar{m},\bar{m}'}$ in eq. (3.23) depend on these additional parameters, which are omitted in (3.23) for brevity.

For g parametrized as in (2.13), $g(\alpha, \beta, \theta; \epsilon_1, \epsilon_2, \delta) = e^{\alpha\sigma_3}(-1)^{\epsilon_1}(i\sigma_2)^{\epsilon_2} g_\delta(\theta) e^{\beta\sigma_3}$, the dependence of the matrix element $K^j_{m,\bar{m}}(g)$ on α and β is

$$K^j_{m,\bar{m}}(g) = k^j_{m,\bar{m},\epsilon_1,\epsilon_2,\delta}(\theta) e^{2i(m\alpha + \bar{m}\beta)} \,. \tag{3.24}$$

Similarly, for $g' \in \mathrm{SU}(2)$ in the parametrization (3.3), $g'(\alpha', \beta', \theta') = e^{i\beta'\sigma_3} e^{i\theta'\sigma_2} e^{i\alpha'\sigma_3}$, the element $D^{j'}_{m',\bar{m}'}(g')$ is of the form

$$D^{j'}_{m',\bar{m}'}(g') = d^{j'}_{m',\bar{m}'}(\theta') e^{2i(m'\beta' + \bar{m}'\alpha')} . \tag{3.25}$$

The operator $V^{j,j'}_{m,m';\bar{m},\bar{m}'}$ is invariant under the gauge transformation (3.4) provided that

$$\begin{aligned} m &= -\bar{m}' \\ m' &= -\bar{m} \end{aligned} \tag{3.26}$$

When this is satisfied, V has the form

$$V^{j,j'}_{m,m';m',m} = k^{j}_{m,m',\epsilon_1,\epsilon_2,\delta}(\theta) d^{j'}_{m',m}(\theta') e^{im(\alpha-\alpha')} e^{im'(\beta-\beta')} , \tag{3.27}$$

which indeed depends only on the four gauge invariant coordinates $\theta, \theta', \alpha - \alpha', \beta - \beta'$. These coordinates parametrize the coset manifold. V also depends on the discrete labels $\epsilon_1, \epsilon_2, \delta$ which denote the different regions in figure 3.

Although in unitary representations of $\mathrm{SL}(2,\mathbb{R})$ the labels m and $\bar{m}$ take any real value, the coset condition (3.26) projects out all SL(2) operators except for those with integral or half integral m and $\bar{m}$, since the SU(2) quantum numbers m' and $\bar{m}'$ are obviously integral or half integral. Geometrically, the half integrality of m and $\bar{m}$ guarantees that the vertex operator V remains single valued after the identifications (3.5).[12] This is also valid along the surfaces on which a single element of g vanishes, where K takes the form (2.40). This form embodies the special properties of that segment of space. In addition, m and $\bar{m}$ are bounded by the requirement

$$|m|, |\bar{m}| \leq j' < \frac{k}{2} , \tag{3.28}$$

where the latter condition comes from unitarity in the SU(2) part of (3.1). We will return to the spectrum of the theory in section 4, where we also discuss excited string states; here we note that (3.28) implies a large depletion in the spectrum of states that arises in going from $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ to the coset (3.1). The spectrum of energies of single particle states[13] is discrete and bounded from above (the energy in the whiskers is given by $m + \bar{m}$ or $m - \bar{m}$ depending on whether λ_+ or λ_- is timelike; see figure 4).

Wavefunctions of the form (3.27) can be used to set up scattering states in the whiskers part of the NW geometry and study their dynamics. Consider, for example, an operator V of the form

$$V^{j,j'}_{m,m';m',m} = \mathrm{U}(\lambda, \mu; j, \epsilon; g) D^{j'}_{m',m}(g') , \tag{3.29}$$

where $\mathrm{U}(\lambda, \mu; j, \epsilon; g)$ is the combination of SL(2) wave functions discussed in eq. (2.41): $\mathrm{U}(\lambda, \mu; j, \epsilon; g) \sim K_{++} - \frac{\sin(\pi\mu)}{\sin(\pi\lambda)} K_{--}$. Here[14] $\lambda = -im - j$ and $\mu = -im' - j$, as in (2.20);

[12] This may play an important role in ensuring that the model is consistent even in the presence of regions with closed timelike curves.

[13] Here we discuss the lowest lying string states; see section 4 for the generalization to excited states.

[14] Here and below we replace $m' \to -m'$ relative to the previous conventions, so that λ and μ are treated symmetrically.

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m and m' are half integer and j is of the form $j = -1/2 + is$ with real s. In region 1 of figure 3, whose geometry is time independent, V describes a combination of an incoming and an outgoing wave from the asymptotic region $\theta \to \infty$. As discussed in the previous section, on the group manifold U was constructed in such a way that no flux enters region 1 from region I. This property is inherited by (3.29) on the coset manifold. The wave (3.29) contains no flux coming from the adjacent cosmological region I in figure 3 through the big crunch of the latter.

Without loss of generality, one can consider the case[15] when λ_- is timelike, and $\mathrm{sign}(s) = \mathrm{sign}(\omega_-)$, where

$$\omega_\pm = \frac{1}{2}(m \pm m') \,, \tag{3.30}$$

so that ω_- is the energy, while s and w_+ are components of spatial momentum. In this case, $R(j; m, m')$ of (2.42) can be thought of as a reflection coefficient for a wave coming from infinity in region 1, and scattering from the big crunch/bang at $\theta = 0$. A part of this wave may penetrate through the big bang into the cosmological region II. Indeed,

$$|R|^2 = \frac{\cosh(2\pi\omega_+) + \cosh(2\pi(s - \omega_-))}{\cosh(2\pi\omega_+) + \cosh(2\pi(s + \omega_-))} \,, \tag{3.31}$$

which implies

$$|R| \leq 1 \,. \tag{3.32}$$

In the coset, the transition from the cosmological regions to the whiskers and vice versa is possible only via the big crunch/bang regions. The part of the incident wave that is not reflected (3.31) apparently makes this transition from the whisker to the cosmological region.

As in section 2, when $m = m'$, generically, the functions K develop a logarithmic divergence at the big crunch/bang of regions I/II, where whisker 1 is connected (similarly, when $m = -m'$ they develop a singularity at a different big crunch/bang, say, the big crunch/bang of regions II/III). However, the combination U is regular, and for the continuous series representations $R(j; m, m)$ is a phase.[16] Therefore, for $m = m'$ the two independent wavefunctions in the continuous series can be chosen to be as follows. One — the combination U – is regular at the singularity, and describes a wave which is fully reflected from it. The other is logarithmically divergent at the singularity, and is not fully reflected.

A peculiar feature of the geometry of the whiskers is that the energy and momentum assignments for the incoming and outgoing waves depend on the value of the coordinate θ'. According to eqs. (3.10), (3.22) and figure 4, for $\theta' > \pi/4$, λ_- serves as a time coordinate for all values of θ; hence the energy and momentum are identified as in (2.43). However, for $\theta' < \pi/4$ the roles of λ_- and λ_+ are reversed once $\cosh 2\theta \cos 2\theta' > 1$. This leads to a discontinuity in the assignment of momenta and energies. The sense of incoming and outgoing waves does not change. As discussed in the previous subsection, the surface (3.22) at

[15] In other cases, the assignments "incoming/outgoing" waves is different.

[16] The reflection coefficient also approaches a phase as s tends to 0.

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which the assignment of energy and momentum flips, corresponds to a timelike singularity in the geometry (3.10)–(3.12). The wavefunctions are regular on this domain wall. It is possible that this is due to the fact that this singularity does not originate from fixed points of any gauge transformation (see the discussion at the end of subsection 2.1).

So far we discussed the scattering process in region 1. Another interesting feature of the NW model is that, as we saw in section 2, the forms of the wavefunction in the different regions in figure 3 are not independent. In particular, specifying the profile (3.29) in region 1 produces some particular, non-trivial profiles of the field in other regions as well. This behavior can be determined by following the combination U given by eq. (2.41) to the various regions, using eqs. (2.26) – (2.37). For instance, in PSL(2) ($\epsilon = 0$), if the incoming wave in region 1 has weight one, the asymptotic behavior of $\mathrm{U}(g_4)$, which follows from eq. (2.37), implies that in region 4 there is an incoming wave with weight

$$-\frac{\sin(\pi\mu)}{\sin(\pi\lambda)}\frac{\Gamma(-1-2j)}{B(\lambda,-\lambda-2j)}\left[\frac{\Gamma(1-\mu)}{\Gamma(-\mu-2j)}+\frac{\Gamma(\mu+2j+1)}{\Gamma(\mu)}\right],$$

and an outgoing wave with weight

$$-\frac{\sin(\pi\mu)}{\sin(\pi\lambda)}\frac{\Gamma(1+2j)}{B(\lambda,-\lambda-2j)}\left[\frac{\Gamma(-\lambda-2j)}{\Gamma(1-\lambda)}+\frac{\Gamma(\lambda)}{\Gamma(\lambda+2j+1)}\right].$$

We see that specifying the boundary conditions near the boundary of region 1 (at infinity) to correspond to the wavefunction U (2.41), actually describes a more complicated process, with incoming and outgoing waves that are in general non-zero in the different whiskers and cosmological regions connecting them.[17] The total incoming flux from all regions equals the outgoing flux. This is guaranteed by current conservation on the group manifold. Since the flux is conserved for every wave on the group, it is also conserved for the restricted values allowed by the coset.

The correlation between the behaviors of the wavefunctions in different whiskers might be interesting for studying the question of possible violations of causality in this background due to the presence of closed timelike curves, since one way to avoid violations of causality is to impose constraints on the Cauchy data in the theory. We see that such constraints arise naturally in the NW model. One might expect that correlations between the different regions will lead to an effective non-locality in the physics seen by any given observer. This is an interesting issue that deserves further study.

Finally, following the discussion in section 2, wavefunctions in the discrete series representations decay exponentially towards the boundaries of the whiskers. They are localized in the compact universes and their vicinity, hence describing states "living" in the expanding and contracting cosmologies. The physics of these states is encoded in correlation functions of the non-normalizable observables with real j familiar from AdS and LST. The principal discrete series states also interact with the scattering waves discussed above in the vicinity of the big bang/crunch, and in the cosmological regions. We will leave a more detailed analysis of these interactions, as well as correlation functions of non-normalizable operators, to future work.

[17] In the covering groups, one may use combinations of waves corresponding to different ϵ's to eliminate the incoming waves in some of the whiskers.

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4. An algebraic analysis of the superstring on the NW background

In this section we describe type-II string quantization in the NW background. We start, as a warmup, with a closely related type-II background,

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times T^6 \,. \tag{4.1}$$

As discussed above, this background is in fact related to the NW spacetime by an $O(2,2,\mathbb{R})$ transformation. The first factor in (4.1) is the two dimensional Minkowski black hole; the second is an $N=2$ minimal model. The levels of SL(2) and SU(2) in (4.1) must be equal, and will be denoted by k.

Consider first the $\mathrm{SL}(2,\mathbb{R})$ factor in (4.1). $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ acts on the $\mathrm{SL}(2,\mathbb{R})$ group element g as

$$g \to e^{\alpha\sigma_3} g e^{\beta\sigma_3} \,, \tag{4.2}$$

which is a spacelike non-compact direction in the group manifold. In the usual conventions (see e.g. [20]), it is generated by the current $(iJ_3, i\bar{J}_3)$, where $J_3 = i \,\mathrm{Tr}\, \sigma_3 g^{-1}\partial g$ satisfies

$$J^3(z) J^3(0) = -\frac{k}{2z^2} \,. \tag{4.3}$$

If one couples this current to a gauge field via

$$\mathcal{L}_{\rm gauge} = i\bar{A}J_3 + iA\bar{J}_3 \,, \tag{4.4}$$

and integrates out the $\mathrm{SL}(2,\mathbb{R})$ degrees of freedom, one finds the effective lagrangian for the gauge field

$$\mathcal{L}_A = \frac{k\pi}{2}\left(\bar{A}\frac{\partial}{\bar\partial}\bar{A} + A\frac{\bar\partial}{\partial}A - 2A\bar{A}\right), \tag{4.5}$$

which is invariant under the gauge transformation $\delta A = \partial\alpha$, $\delta\bar{A} = \bar\partial\alpha$. Imposing the gauge fixing condition $\partial_a A^a = 0$, one can parametrize the gauge field via a scalar field t,

$$A = i\partial t; \quad \bar{A} = -i\bar\partial t \,. \tag{4.6}$$

Plugging this into (4.5), we see that t is dynamical, and the full system consists of three parts:

$$\mathcal{L} = \mathcal{L}_{\mathrm{SL}(2)} + \mathcal{L}_t + \mathcal{L}_{\rm ghost} \,, \tag{4.7}$$

where (after rescaling t to make it canonically normalized)

$$\mathcal{L}_t = -\partial t \bar\partial t \,. \tag{4.8}$$

Note that t is a timelike coordinate. Since the gauge group (4.2) is non-compact, t is non-compact as well.

$\mathcal{L}_{\rm ghost}$ is the ghost lagrangian

$$\mathcal{L}_{\rm ghost} = b\bar\partial c + \bar{b}\partial\bar{c} \,. \tag{4.9}$$

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The left-moving ghosts b, c have scaling dimensions $\Delta = 1, 0$, respectively, and similarly for the right movers $\bar{b}$, $\bar{c}$.

The system (4.7) has a BRST symmetry generated by

$$Q_{\mathrm{BRST}} = \oint \frac{dz}{2\pi i} c \left(iJ_3 + i\sqrt{\frac{k}{2}} \partial t \right) + \oint \frac{d\bar{z}}{2\pi i} \bar{c} \left(i\bar{J}_3 + i\sqrt{\frac{k}{2}} \bar{\partial} t \right). \tag{4.10}$$

Physical states belong to the cohomology of Q_{BRST}.

So far we have discussed bosonic CFT on $\mathrm{SL}(2)/\,\mathrm{U}(1)$. In the fermionic string there are also worldsheet fermions in the adjoint representation of $\mathrm{SL}(2)$, and the $\mathrm{U}(1)$ that one gauges is a *super* affine Lie algebra generated by the supercurrent

$$(\psi_3 + \theta J_3, \bar{\psi}_3 + \bar{\theta} \bar{J}_3)\,, \tag{4.11}$$

where J_3 is the total $\mathrm{U}(1)$ current (it receives a contribution from the fermions, $\psi^+\psi^-$), and k is the level of the full $\mathrm{SL}(2)$ (which decomposes into bosonic and fermionic contributions, via $k = (k+2)_B + (-2)_F$). There are also bosonic ghosts (β, γ), $(\bar{\beta}, \bar{\gamma})$, each with $\Delta = 1/2$, associated with gauging $i\psi_3$; t has a superpartner ψ_t.

The BRST charge (4.10) receives a contribution

$$\oint \frac{dz}{2\pi i} \gamma \left(i\psi_3 + i\sqrt{\frac{k}{2}} \psi_t \right) + \oint \frac{d\bar{z}}{2\pi i} \bar{\gamma} \left(i\bar{\psi}_3 + i\sqrt{\frac{k}{2}} \bar{\psi}_t \right). \tag{4.12}$$

In order to describe the string theory on (4.1) we have to combine $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ with the other factors, $\mathrm{SU}(2)/\,\mathrm{U}(1)$ and T^6. $\mathrm{SU}(2)/\,\mathrm{U}(1)$ is an $N = 2$ minimal model, but for our purposes it is convenient to describe it in a way similar to $\mathrm{SL}(2)/\,\mathrm{U}(1)$ above. One starts with $\mathrm{SU}(2)$, and introduces a gauge field which couples to the Cartan generator K_3, and gives rise, as in (4.6), to a scalar superfield (X, ψ_x), which is compact (like the gauge symmetry it is associated with). There are also new ghosts, (b', c') and (β', γ'), which are analogs of (4.9), (4.10), (4.12).

To study type-II string theory on (4.1), one also has to apply a chiral GSO projection. While the background (4.1) has $N = 2$ superconformal symmetry on the worldsheet, it does not seem to be spacetime supersymmetric, since the $\mathrm{U}(1)_R$ charges are not integer (they are imaginary). A natural proposal for the action of GSO is: $\psi \to -\psi$ for all fermions (those associated with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times T^6$, as well as ψ_t, ψ_x) and $(\beta, \gamma, \beta', \gamma') \to -(\beta, \gamma, \beta', \gamma')$ for the bosonic ghosts. The question of spacetime SUSY in this language is the question whether there exists a holomorphic, dimension $(1, 0)$ operator, $J_\alpha(z)$, which can be used to form a spacetime supercharge

$$Q_\alpha = \oint \frac{dz}{2\pi i} J_\alpha(z)\,. \tag{4.13}$$

A natural conjecture here would be

$$J_\alpha(z) = e^{-\frac{\varphi}{2}} S_\alpha e^{-\frac{\varphi_1}{2} - \frac{\varphi_2}{2}}\,, \tag{4.14}$$

where φ, $\bar{\varphi}$ are the bosonized superconformal ghosts, $S_\alpha(z)$ is a spin field for the fourteen worldsheet fermions associated with $\mathrm{SL}(2)\times\mathrm{SU}(2)\times T^6$, and ψ_x, ψ_t, and φ_1, φ_2 are related to the bosonic ghosts β, γ, β', γ' via

$$\beta\gamma = \partial\varphi_1\,; \qquad \beta'\gamma' = \partial\varphi_2\,. \tag{4.15}$$

The total dimension of J_α is

$$\Delta(J_\alpha) = \frac{3}{8} + 7\times\frac{1}{8} - 2\times\frac{1}{8} = 1\,, \tag{4.16}$$

as needed for spacetime SUSY, but while (4.14) is invariant under (4.12) (this imposes some constraints on the spinor index α), it is not invariant under (4.10) (since all J_α are charged under the CSA generator of $\mathrm{SL}(2,\mathbb{R})$, J_3). Hence, the spacetime theory is not supersymmetric, which is not unexpected in a time-dependent background. We have not proven that no other supercurrents $J_\alpha(z)$ exist, but it is natural to expect that this is indeed the case.

We next discuss some aspects of the resulting low energy spectrum. Due to the chiral GSO projection, the lowest lying states in the spectrum are "gravitons." In the $(-1,-1)$ picture they have vertex operators

$$e^{-\varphi-\bar{\varphi}}V_{j;im,im}e^{i\sqrt{\frac{2}{k}}mt}V'_{j';m',m'}e^{\sqrt{\frac{2}{k}}m'X}e^{i\vec{k}\cdot\vec{y}}\xi_{\mu\nu}\psi^\mu\bar{\psi}^\nu\,, \tag{4.17}$$

where the notation is as follows. As before, φ, $\bar{\varphi}$ are the bosonized superconformal ghosts. $V_{j;im,i\bar{m}}$ is a primary of $\mathrm{SL}(2)_L\times\mathrm{SL}(2)_R$ affine Lie algebra, with scaling dimension $\Delta = \bar{\Delta} = -j(j+1)/k$. $V'_{j';m',\bar{m}'}$ is a primary of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ affine Lie algebra, with scaling dimension $\Delta = \bar{\Delta} = j'(j'+1)/k$. The exponentials of t and X are the dressing by the gauge fields. Gauge invariance under (4.10) and its SU(2) analog implies $m = \bar{m}$ and $m' = \bar{m}'$. $\xi_{\mu\nu}$ is a polarization tensor, with μ,ν running over the six directions along the T^6, and the "coset directions," $\psi^\pm$ in SL(2) and $\chi^\pm$ in SU(2). For polarizations in the "coset" directions $\mu = \pm,\pm'$, the values of im in (4.17) are actually shifted by ±1, so that the eigenvalue of the total J^3 on the state is imaginary. This can be seen by imposing the standard transversality conditions on the vertex operator (4.17). Thus, for example, the $\psi^+\bar{\psi}^+$ term in (4.17) in fact looks like

$$e^{-\varphi-\bar{\varphi}}V_{j;im-1,im-1}e^{i\sqrt{\frac{2}{k}}mt}V'_{j';m',m'}e^{\sqrt{\frac{2}{k}}m'X}e^{i\vec{k}\cdot\vec{y}}\xi_{++}\psi^+\bar{\psi}^+\,. \tag{4.18}$$

The mass-shell condition satisfied by (4.17) is:

$$-\frac{j(j+1)}{k} - \frac{m^2}{k} + \frac{j'(j'+1)}{k} - \frac{m'^2}{k} = 0\,. \tag{4.19}$$

The right-movers give rise to a similar equation. One can think of m as the energy of the resulting state. Equation (4.19) has two kinds of solutions. For small enough m, the solution for j is real. In this case, one of the solutions of (4.19) corresponds to a non-normalizable wave function (4.17), which is exponentially supported at the "boundary" of SL(2)/U(1), the region far from the horizon of the black hole. The other solution is normalizable, and describes a principal discrete series state.

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For energies m larger than a critical value that depends on j', m', the solution of (4.19) has the form $j = -1/2 + is$, and the corresponding wave function is delta function normalizable. It can be thought of as describing a scattering state of a graviton (or dilaton, or NS B field) coming in from infinity and scattering from the black hole, as discussed in [22] and in the previous sections.

The non-normalizable observables (4.17) with $j \in \mathbb{R}$, give rise to non-fluctuating couplings, or superselection sectors in the worldsheet lagrangian [36]. In analogy with the case of $SL(2,\mathbb{R})(=AdS_3)$ and Little String Theory, one is led to interpret them as off-shell observables in a dual theory, and the region at infinity (the analog of (3.13)) as a holographic screen (as in [16]).[18]

One can also discuss excited string states in the same way. At oscillator level N, the mass-shell condition (4.19) is replaced by

$$-\frac{j(j+1)}{k} - \frac{m^2}{k} + \frac{j'(j'+1)}{k} - \frac{m'^2}{k} + N = 0\,. \tag{4.20}$$

The qualitative picture is the same: below a certain critical energy m_0 which depends on N, one finds non-normalizable observables with $j \in \mathbb{R}$ and principal discrete series states. Above that critical value, one has scattering states with $j \in -1/2 + is$. Using standard techniques, one can compute scaterring amplitudes of the states with $j \in -1/2 + is$ and correlation functions of the sources with $j \in \mathbb{R}$.

We now move on to a discussion of the NW background (3.1), (3.2). The two U(1) symmetries that one gauges are in this case generated by

$$(iJ_3, \bar{K}_3); \;\; (K_3, i\bar{J}_3)\,. \tag{4.21}$$

One can repeat the analysis of the two dimensional black hole above for this theory. The BRST charge (4.10) is replaced by

$$Q_{\rm BRST} = \oint \frac{dz}{2\pi i} c \left(iJ_3 + i\sqrt{\frac{k}{2}}\partial t \right) + \oint \frac{d\bar{z}}{2\pi i} \bar{c} \left(\bar{K}_3 + i\sqrt{\frac{k}{2}}\bar{\partial} t \right), \tag{4.22}$$

and similarly, for the other U(1) in (4.21) one has

$$Q_{\rm BRST} = \oint \frac{dz}{2\pi i} c \left(K_3 + i\sqrt{\frac{k}{2}}\partial t' \right) + \oint \frac{d\bar{z}}{2\pi i} \bar{c} \left(i\bar{J}_3 + i\sqrt{\frac{k}{2}}\bar{\partial} t' \right). \tag{4.23}$$

The analog of the graviton (4.17) is now

$$e^{-\varphi-\bar{\varphi}} V_{j;im,im'} V'_{j';m',m} e^{i\sqrt{\frac{2}{k}}(mt+m't')} e^{i\vec{k}\cdot\vec{y}} \xi_{\mu\nu}\psi_\mu\bar{\psi}_\nu\,, \tag{4.24}$$

we have implemented the gauge conditions $iJ_3 + \bar{K}_3 = 0$, $K_3 + i\bar{J}_3 = 0$, and the physical state condition (4.19) is implied. j runs over the principal continuous series, $j = -1/2 + is$, $s \in \mathbb{R}$, and over the principal discrete series, $j \in \mathbb{R}$, $-\frac{1}{2} < j < \frac{1}{2}(k-1)$ for normalizable states [37].

[18] The correlation functions of the observables (4.17) with j real are complex. It would be interesting to understand the implications of this better.

An important difference between (4.24) and (4.17) is that in (4.24) m, m' run over a finite set of values, $m, m' \in \frac{1}{2}\mathbb{Z}$, $|m|, |m'| \leq j' \leq \frac{1}{2}k-1$. Thus, unlike (4.17), which describes an infinite number of states, in (4.24) there is a finite number of physical states. This is interesting from the point of view of holography. In the black hole case, the fact that the energy m in (4.17) is continuous, signals the fact that the dual theory is $0+1$ dimensional (i.e., it is quantum mechanics). For NW, the finite number of observables (4.24) seems to suggest that the dual theory is zero dimensional (perhaps a finite collection of points or a topological theory).

At higher oscillator levels N, one still has a finite number of observables at each level, but that number is larger the higher the level. Given N, the eigenvalue of the SU(2) generator K_3 is bounded by $|K_3| \leq j' + N$. Hence, following the same reasoning as above, the SL(2) quantum number m is bounded by $|m| \leq j' + N \leq \frac{k}{2} - 1 + N$. Thus, for given N, the number of states is finite.

Correlators of the operators discussed above can be computed by using standard perturbative worldsheet techniques. An important part of the calculation comes from the operators $V_{j;im;im'}$ from the underlying SL(2) theory. For instance, the reflection coefficient (2.44) is given by a two point function on the sphere (see [23, eq. (3.6)][19]):

$$\begin{aligned} R(j;m,m';k) &\equiv \langle V_{j;-im;-im'} V_{j;im;im'} \rangle = \\ &= \frac{\Gamma(1-\frac{2j+1}{k})\Gamma(-2j-1)\Gamma(j+1+im)\Gamma(j+1-im')}{\Gamma(1+\frac{2j+1}{k})\Gamma(2j+1)\Gamma(-j+im)\Gamma(-j-im')} \,. \end{aligned} \tag{4.25}$$

In the semiclassical limit, $k \to \infty$, this correlator coincides with the reflection coefficient $R(j;m,m')$ given in eq. (2.44); (4.25) is valid for all k. Note that, since in the continuous series $j = -1/2 + is$, the factor $\Gamma(1-\frac{2j+1}{k})/\Gamma(1+\frac{2j+1}{k})$ is a phase. Hence, the $1/k$ corrections only add a j dependent phase to the semiclassical reflection coefficient, and in particular they do not modify (3.31).

It is in principle possible to use the algebraic description to study interactions. In particular, one can use the results of section 4 in [23] to compute the tree level three point functions. These seem to be regular for imaginary values of the spacelike Cartan eigenvalues.

5. Summary

The main purpose of this paper was to study in more detail the NW model [2], which describes a closed cosmology starting with a big bang and ending at a big crunch.

Our attitude to this problem was that since the model can be described as a coset CFT, string propagation in this spacetime should make sense, and thus it can be used to study various conceptual and quantitative issues that arise in backgrounds with cosmological (big bang and big crunch) singularities. Examples include the nature of observables in such spacetimes, the question whether one should continue past such singularities to pre big bang and post big crunch regimes, and if so how solutions of the wave equations are

[19] We choose $\nu(k) = 1$ in [23, eq. (3.6)] (see [38] for a discussion on the freedom to make such a choice).

matched across the singularity. A better understanding of these issues is necessary for studying the interactions between different regions separated by cosmological singularities, and of the question whether the existence of singularities necessarily leads to large quantum effects (i.e. breakdown of string perturbation theory).

Our main results are the following.

String theory on the coset spacetime describes a sequence of big bang/big crunch universes attached to each other at the singularities in the way indicated in figure 3. Additional non-compact regions are attached to the sequence of closed universes, at the big bang and big crunch singularities as well. These non-compact regions, which we referred to as whiskers, are static and have a boundary at infinity, near which they look like spacelike linear dilaton solutions.

The observables of the theory are defined by studying the behavior of the fields near the boundary, as discussed in [16] in the context of Little String Theory. As in LST, there are two kinds of observables. One corresponds to scattering states incident on the geometry from the boundary; the other corresponds to non-normalizable wavefunctions supported near the boundary. Both kinds of observables provide information about the physics in the bulk of spacetime, and in particular on the closed cosmological region. The non-normalizable wavefunctions correspond to states localized in the compact universes and their vicinity.

The scattering states, which correspond to vertex operators in the principal continuous series of $SL(2,\mathbb{R})$, are partially reflected from the cosmological singularity and partially transmitted into the cosmological region. The reflection coefficient for this process is given in eq. (3.31). This provides an example of non-trivial interaction between regions separated by cosmological singularities.

The non-normalizable observables seem to give rise to the analog of off-shell Green functions in AdS space and LST. These Green functions encode the local dynamics in the bulk of spacetime. The two point function of these observables is given by the standard SL(2) result (see e.g. [23, eq. (3.6)]). Higher point functions of both kinds of observables can in principle be computed as well using results from CFT on $SL(2,\mathbb{R}) \times SU(2)$.

The question of matching of solutions of wave equations across cosmological singularities is important for applications (see e.g. [9, 10]) and is in general ambiguous in the framework of QFT in curved spacetime. On the other hand, here the description of the space as a coset of an underlying $SL(2,\mathbb{R})$ manifold provides an organizing principle that allows one to continue the wavefunctions past the singularities. In section 2 we described the wavefunctions in $SL(2,\mathbb{R})$, and in section 3 we explained how they give rise to uniquely determined wavefunctions in all the regions of the extended NW spacetime.

We found that the spectrum of the theory is significantly depleted compared to other related examples. We presented evidence that this depletion is related to the appearance of closed timelike curves in some regions of the extended NW spacetime (the whiskers). It would be interesting to obtain a better understanding of this relation, and in particular of the (lack of) violations of causality in this model.

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Many questions deserve further study. We argued that three point functions on the sphere should be finite, using the results of [23]. It would be interesting to establish this in more detail. It would also be important to understand the structure of four and higher point functions, and in particular the effects of the cosmological singularities on these correlation functions.

Since the model is not supersymmetric, it is of interest to consider loop corrections, and in particular the question of stability of the classical spacetime against quantum corrections. Other issues that require better understanding are the role of the timelike singularity (3.22) in the whiskers, the effect of the closed timelike curves in the whiskers, and the local physics in the closed cosmological regions.

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